

A complete axiomatization of timed bisimulation for

- The countable ordinals $(\omega, +, \cdot)$ —

$$\begin{array}{l}
\text{S} \quad E + F = F + E \\
\text{SF} \quad E + (F + G) = (E + F) + G \\
\text{S} \quad E + E = E \\
\text{S} \quad E + \mathbf{0} = E \\
\\
\text{TD} \quad \varepsilon(t).(E + F) = \varepsilon(t).E + \varepsilon(t).F \\
\text{TA} \quad \varepsilon(t + u).E = \varepsilon(t).\varepsilon(u).E \\
\text{T} \quad \varepsilon(\cdot).E = E \\
\\
\text{R} \quad \mathbf{fix}(x = E) = E\{\mathbf{fix}(x = E)/x\}
\end{array}$$

RF If $F = E\{F/x\}$, then $F = \mathbf{fix}(x = E)$ provided x is action guarded in E

FIG RE F – The axiom system \mathcal{G}

$$\begin{array}{l}
\text{MP} \quad \tau.E + \varepsilon(c).F = \tau.E \\
\text{AP} \quad a.E + \varepsilon(t).a.E = a.E \\
\text{NP} \quad \varepsilon(t).\mathbf{0} = \mathbf{0}
\end{array}$$

FIG RE – The axiom system \mathcal{F} is \mathcal{G} plus MP, AP and NP

$$\begin{array}{l}
\text{MP} \quad \tau.E + \varepsilon(c).F = \tau.E \\
\text{P} \quad E + \varepsilon(t).E = E
\end{array}$$

FIG RE – The axiom system \mathcal{E} is \mathcal{G} plus MP and P

Then $E \sim F$ iff for all vectors $P = P_1, \dots, P_m$ $E\{P/x\} \sim F\{P/x\}$.

PROPOSITION 5 – **THEOREM 5** – Timed bisimulation equivalence forms a congruence over **TC**.

In the remainder of this paper we shall present a complete axiomatization of \sim over **TC**.

3 Axiomatization and soundness

In various equational laws were proved to hold for Yang Yi's timed CCS modulo timed bisimulation equivalence and in \mathcal{F} a set of such axioms was shown to be complete over the language of recursion free **TC** processes with delays from the time domain of the positive reals. We shall now present an axiomatization which will be proven complete for \sim over the whole of **TC** i.e. complete for regular process expressions with action guarded recursion. The detailed proof of completeness occupies Section 4 of this paper.

Yang's axiomatization for recursion free **TC** processes is given by the axiom system \mathcal{F} in Figures F and our axiomatization for regular **TC** process expressions is given by the axiom system \mathcal{E} in b – d e – x u – 5

4 Completeness

In this section we shall present the proof of completeness of the set of laws \mathcal{E} over **TC**. The structure of the proof of this result will follow closely the most beautiful arguments used by Milner in [Mil84] to prove the completeness of the axiomatizations for strong bisimulation and observational congruence over regular CCS processes.

The structure of the completeness proof will be as follows. First of all we shall show that every **TC** expression E provably satisfies a certain kind of equation set. This is what Milner calls the *Equational Characterization Theorem*. Next we shall show that if $E \sim F$ and E provably satisfies an equation set which F provably satisfies another equation set, then both E and F provably satisfy a common equation set. Finally we show that whenever two **TC** expressions provably satisfy the same equation set, then \mathcal{E} proves that they are equal.

DEFINITION— An equation set $x = E$ is a finite non-empty sequence of declarations $x = E_1, \dots, x_n = E_n$, where the x_i s are pairwise distinct variables, and the E_i s are **TC** expressions.

A vector $F = F_1 \dots F_n$ satisfies $x = E$ iff $\forall i. F_i \sim E_i\{F/x\}$.

For an equational theory \mathcal{T} , a vector $F = F_1 \dots F_n$ \mathcal{T} -provably satisfies $x = E$ iff $\forall i. \mathcal{T} \vdash F_i = E_i\{F/x\}$.

An expression E (\mathcal{T} -provably) satisfies $x = F$ iff we can find a vector E which (\mathcal{T} -provably) satisfies $x = F$ and $E \sim E$ ($\mathcal{T} \vdash E = E$).

We refer to x as the leading variable of the equation set $x = F$.

For example the equation set

$$x = \varepsilon(\dots)$$

$$= E$$

and

$$\begin{aligned}
 \mathcal{G} \vdash E & \\
 = F \{E/w\} & \\
 = H \{F/x\} \{E/w\} & \\
 = H \{E/w\} \{F\{E/w\}/x\} & \quad \text{Propn 5} \\
 = H \{E/w\} \{E/x\} & \\
 = H \{E/x\} & \quad w \notin \text{fv}H
 \end{aligned}$$

and so

$$\begin{aligned}
 \mathcal{G} \vdash E_i & \\
 = F_i \{E/w\} & \\
 = H_i \{F/x\} \{E/w\} & \\
 = H_i \{E/w\} \{F\{E/w\}/x\} & \quad \text{Propn 5} \\
 = H_i \{E/w\} \{E/x\} & \\
 = H_i \{H \{E/x\}/w\} \{E/x\} & \quad \text{above} \\
 = H_i \{H/w\} \{E/x\} & \quad \text{Propn 5} \\
 = G_i \{E/x\} &
 \end{aligned}$$

Thus we have found a standard $x = G$ which $E \mathcal{G}$ provably satisfies. \square

Theorem 5 shows that every expression E in $\mathbf{TC} \mathcal{G}$ provably satisfies a standard equation set $x = G$. The second stepping stone towards the promised completeness theorem is a result showing that if $E \sim F$ where $F \mathcal{G}$ provably satisfies a standard equation set $y = H$, then there re re re re t

Thus each s_{u_i} and of $G_{i'}\{H/z\}$ can be absorbed into $H_{i'}$ and by S S

$$\mathcal{E} \vdash H_{i'} = H_{i'} + G_{i'}\{H/z\} \quad \text{S}$$

We now show that the converse also holds namely that $H_{i'}$ can be absorbed into $G_{i'}\{H/z\}$ —To this end by S and S it is sufficient to prove that each s_{u_i} and of $F_i\{E/x\}$ can be absorbed into $G_{i'}\{H/z\}$ —Again we distinguish three cases depending on the form the s_{u_i} takes—

For any $i \in \mathcal{R}$ and $j \in \mathcal{J}$ either

- $t_j \leq u_k$ for every $k \in \mathcal{K}$ or
- there exists $k \in \mathcal{K}$ such that $t_j > u_k$ —

We proceed to show that in either case

$$\mathcal{E} \vdash G_{i'}\{H/z\} = G_{i'}\{H/z\} \quad \text{E}$$

$$\begin{aligned}
&= H_{i'} + G_{i'}\{H/z\} && 5 \\
&= E_i + G_{i'}\{H/z\} && \cdot \\
&= F_i\{E/x\} + G_{i'}\{H/z\} && \bullet \\
&= G_{i'}\{H/z\}
\end{aligned}$$

Thus $H \mathcal{E}$ provably satisfies $z = G$ and $\mathcal{E} \vdash E = E = H$ so $E \mathcal{E}$ provably satisfies $z = G$. Similarly $E' \mathcal{E}$ provably satisfies $z = G$. \square

The main ingredient of the proof of completeness is a result showing that every standard equation set has a unique solution up to provable equality.

THEOREM (UNIQUENESS SOLUTION) – *If $x = H$ is a standard equation set, then there is a TC expression E which \mathcal{E} -provably satisfies it. Moreover, if another TC*

$$\begin{aligned}
&= \sum_{i \in I} \varepsilon(t_i) \cdot \mu_i \cdot P_i + \varepsilon(t + t_i) \cdot \mu_i \cdot P_i && t + (u - t) = u \\
&= \sum_{i \in I} \varepsilon(t_i) \cdot \mu_i \cdot P_i + \varepsilon(t) \cdot \varepsilon(t_i) \cdot \mu_i \cdot P_i && \text{TA} \\
&= \sum_{i \in I} \varepsilon(t_i) \cdot \mu_i \cdot P_i + \sum_{i \in I} \varepsilon(t) \cdot \varepsilon(t_i) \cdot \mu_i \cdot P_i && S, SF \\
&= \sum_{i \in I} \varepsilon(t_i) \cdot \mu_i \cdot P_i + \varepsilon(t) \cdot \sum_{i \in I} \varepsilon(t_i) \cdot \mu_i \cdot P_i && \text{TD NP} \\
&= P + \varepsilon(t) \cdot P && \bullet
\end{aligned}$$

Thus \mathcal{F} can show any closed instantiation of axiom P – \square

Note that throughout the above proof we have been careful not to assume that the monoidal operation $+$ on the time domain is commutative—Although this is true for most of the examples of the time domain one encounters in the literature it does not hold for e.g. the time domain of the countable ordinals $(\omega, +, \cdot)$ —

6 Concluding remarks

In this paper we have presented a complete axiomatization of timed bisimulation equivalence over open terms with finite state recursion in a generalization of the regular subcalculus of timed CCS—Our inference system for timed bisimulation equivalence is obtained by combining an improved version of our regular subcalculus with standard laws for recursive y defined processes—The proof of completeness of the proposed axiomatization uses an adaptation of Milner's classic arguments presented in [Mil80]—

In the

–Høyer, K-G-Larsen and Wang Yi– Deciding properties of regular related processes–
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Aalborg University Centre. – An extended abstract appeared in the *Proceedings of
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