Calculus in coinductive form

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Abstract

Following these ideas, we introduce in sections 1 and 2 a formal setting for studying and implementing analytic structures by coalgebraic methods. Section 3 proceeds from our stream algebras of Taylor coefficients to derive an abstract characterisation of a different analytic method: Laplace transform. We show that it also arises, like Taylor series, as a coalgebra homomorphism induced by specific stream operations. We compute them and derive the corresponding integral expressions. As a byproduct of the coalgebraic treatment, we obtain a simple characterisation of the Laplace duals of analytic functions.

1 Stream algebras

Our main tool are the fixpoints of functors in the form $\Sigma \times (-) : \mathsf{Set} \to \mathsf{Set}$.

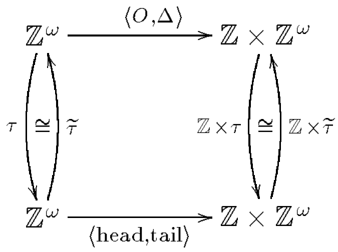
Definition 1.1 *Let* Σ *be a set. A* Σ-stream algebra *is a set* A *together with an isomorphism*

$$\Sigma \times A \xrightarrow[\cong]{c}$$

where

$$\begin{array}{rcl} \Delta\alpha & = & [\alpha_1 - \alpha_0,\ \alpha_2 - \alpha_1,\ \alpha_3 - \alpha_2, \ldots], \\ a + \Sigma\beta & = & [a,\ a + \beta_0,\ a + \beta_0 + \beta_1, \ldots] \end{array}$$

Essentially employing the commutativity of $\mathbb{Z}$, one finds that this stream algebra structure on $\mathbb{Z}^\omega$ is actually isomorphic with (4), via



The n-th entries of the sequences $\tau(\alpha)$, resp. $\widetilde{\tau}(\alpha)$, are defined:

$$\begin{array}{rcl} \tau(\alpha)_n & = & \sum_{i=0}^{n} \binom{-n}{i} \alpha_i \\ \widetilde{\tau}(\alpha)_n & = & \sum_{i=0}^{n} \binom{n}{i} \alpha_i \end{array}$$

Note that τ is actually the discrete Taylor transformation,

In general, every first order initial value problem involving only analytic functions can be solved in this fashion [2, thms. 4.4–4.5], as well as many important higher order linear differential equations [3, ch. 10]. The recurrence relations on the coefficients tend to be tedious, though, and extracting the actual recursive formulas for them is not always feasible.

Other analytic methods are captured by different stream algebras and stream algebra homomorphisms between them.

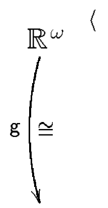
3 Laplace transform

3.1 Rings of streams

Now consider the algebra

$$\begin{array}{rlcl}
\text{head}: & \mathbb{R}^\omega & \longrightarrow & \mathbb{R} \\
 & \alpha & \longmapsto & \alpha_0 \\[1ex]
\mathsf{tail}: & \mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\
 & \alpha & \longmapsto & [\alpha_1,\ 2\alpha_2,\ 3\alpha_3, \ldots] \\[1ex]
\mathsf{cons}: & \mathbb{R}\times\mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\
 & \langle a, \beta\rangle & \longmapsto & [a,\ \beta_0,\ \frac{\beta_1}{2},\ \frac{\beta_2}{3}, \ldots]
\end{array}$$

This is yet another version of the stream algebra of infinite lists of numbers, isomorphic with the "original" via



3.3 La lace transform of analytic functions

Recall that Laplace transform $\mathcal{L}$ takes a real locally integrable function $f(x)$ and returns a function

Corollary 3.4 *A real function f is analytic at 0 if and only if its Laplace dual $\mathcal{L}\{f\}$ is a coanalytic function vanishing at ∞. Laplace transform is a bijection $\mathcal{L} : \mathbb{A} \longrightarrow \mathbb{H}_\infty$.*

But it is not a mere bijection: as a morphism it is completely determined by its preservation properties:

Corollary 3.5 *Laplace transform*